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On the uniqueness of the representation of a convex polygon by its Hough transform

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Abstract

In this paper we prove the uniqueness of the representation of a convex n-gon (bounded or unbounded for $n \geqslant 5$, bounded for $n \geqslant 3$) by the peaks of its Hough Transform.

Keywords: Convex polygons; Hough transform

1. Introduction

The most popular method for the recognition of straight lines in a digital image is the Hough Transform (Hough, 1962; Ballard, 1981). The main idea of the method is to solve the straight line equation

$$y = cx + m \tag{1}$$

for m:

$$m = (-x)c + y. \tag{2}$$

The new equation defines a straight line in the parameter (Hough) space. This straight line has a slope equal to $-x$ and an intercept equal to y where (x, y) are the coordinates of a point. According to this equation a point in the image space is mapped on a straight line in the parameter space. This straight line represents the set of lines that belong to the pencil that passes through the point (x, y). The parameter space is segmented and each feature point (x, y) votes for the lines that pass through it. Highly voted lines given an indication of the existence of straight line segments.

The efficiency of the Hough Transform in recognizing digital straight line segments makes the technique particularly well suited for the recognition of polygons in a digital image. A lot of research has been done in this area during the past few years (Engelbrecht and Wahl, 1988; Turney et al., 1985; Rosenfeld and Weiss, 1995). In this paper we address the problem of the uniqueness of the representation of a convex polygon by the peaks of its Hough Transform. We will show that a bounded n-gon (with $n \geqslant 3$) is uniquely determined by the spikes of the Hough Space. We will also show that a convex n-gon (with $n \geqslant 5$) is uniquely determined by the spikes it gives in the Hough Space regardless of whether it is bounded or unbounded. We will assume that no digitization errors exist and that the peaks give the parameters of the equation of a straight line with no error.

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In Section 2 we present the background work and the notation that will be used throughout this paper. Section 3 presents the main body of this paper. In this section we prove that a bounded convex n-gon is uniquely determined by the peaks in the Hough space, and that any convex n-gon, for $n \geqslant 5$, bounded or unbounded, is defined uniquely by the peaks it gives in the Hough space. Finally, Section 4 contains the conclusions.

2. Background–notation

Suppose a straight line L and a convex figure Q (bounded) are given. There are four possible arrangements (see Lyusternik, 1966, p. 8):
Case 1: L and Q have no points in common (line L_1 in Fig. 1),
Case 2: L and Q have one point in common (line L_2 in Fig. 1),
Case 3: L and Q have as an intersection a line segment belonging entirely to the boundary of Q (line L_3 in Fig. 1),

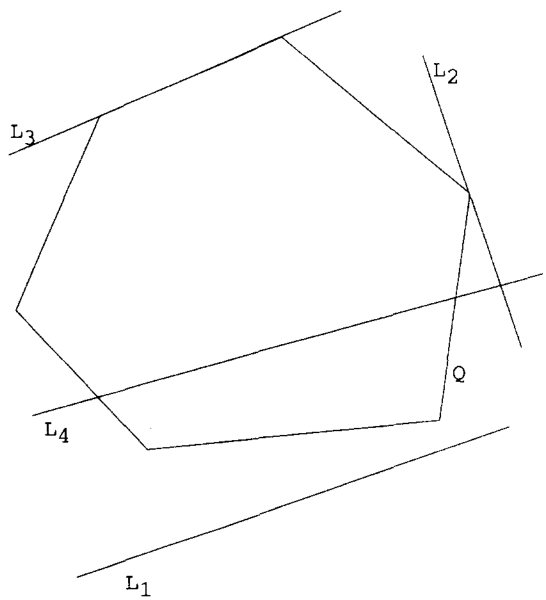

Fig. 1. Illustration of the four different relative positions of a straight line with a convex shape: line L_1 has no common points with the convex shape Q, line L_2 has a single common point with the boundary of Q, line L_3 has an edge segment belonging to the boundary of Q and line L_4 has an intersection lying entirely inside Q except for its endpoints.

Case 4: The intersection of L and Q lies entirely (with the exception of its endpoints) inside Q (line L_4 in Fig. 1).

The following properties will be useful as we develop our theory:
- a straight line cuts the plane into two semi-planes which are convex sets,
- if the intersection of two convex figures is not the empty set it is convex.

When the notation $f(A, B, \ldots, P) > 0$ is used, it is meant that function f becomes positive for all points $A, B, \ldots, P$. We will denote straight lines by L_i and their equations by $f_i(x, y) = y - c_i x - m_i = 0$. The notation $P_0 P_1 \ldots P_{n-1}$ denotes the set of vertices of a bounded convex n-gon as one travels its boundary counterclockwise (see Fig. 2). The notation $L_0 P_0 P_1 \ldots P_{n-2} L_{n-1}$ denotes an unbounded convex n-gon as one travels its boundary counterclockwise (starts from ray $L_0 P_0$ and ends with ray $P_{n-2} L_{n-1}$, see Fig. 3).

3. The uniqueness of the Hough Transform of convex polygons

In order to prove the main theorem that will help us to establish the uniqueness of the representation of convex polygons by their Hough Transform Lemmas 1, 2 and 3 will be needed.

Lemma 1. *Suppose $P_0 P_1 \ldots P_{n-1}$ is a bounded convex n-gon. A straight line L_n, which has a common point with the interior of the n-gon and does not pass through any of its vertices, cuts it into two bounded convex polygons. The number of sides of the two polygons m_1 and m_2 satisfy the following relation*:
$m_1 + m_2 = n + 4.$

Proof. The convexity of the two polygons is guaranteed by the argument that they are the intersection of two convex sets (the original convex n-gon and the two semi-planes, from Section 2 a semi-plane is a convex set). The boundedness is guaranteed by the fact that they both are subsets of the original n-gon. It only remains to show that the relation between the number of the sides of the two polygons which was given in the statement of Lemma 1 is indeed correct. Suppose that one semi-plane of line L_n contains k

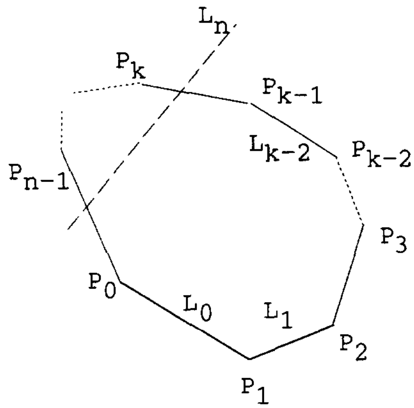

Fig. 2. A bounded convex n-gon which is cut by line L_n. Obviously k vertices $P_0 \ldots P_{k-1}$ are left on one semi-plane and the remaining $(n-k)$ vertices of the n-gon are on the other semi-plane.

vertices of the n-gon and the other $n-k$, where $1 \leqslant k \leqslant n-1$. Obviously, the n-gon is cut into two polygons, one $(k+2)$-gon and one $(n-k+2)$-gon (see Fig. 2). So the sum of the number of vertices of the two polygons is

$$m_1 + m_2 = (k+2) + (n-k+2) = n+4. \quad \square$$

Lemma 2. *Given an unbounded convex n-gon, a straight line L_n which has a common point with its interior and does not pass through any of its vertices either*

(i) *cuts it into an unbounded convex m_1-gon and a bounded convex m_2-gon, which satisfy the relation $m_1 + m_2 = n + 4$, or*
(ii) *cuts it into two unbounded convex polygons which satisfy the relation $m_1 + m_2 = n + 3$.*

Proof. The two cases are shown in Fig. 3 (lines L_n (i) and L_n (ii)). The proof is in the same spirit as in Lemma 1. $\square$

Lemma 3. *Suppose a line L_n cuts a convex polygon into two. Supppose also that none of the vertices of the polygon belongs to the straight line L_n. The two new polygons that are created have m_1 and m_2 sides. The number of sides of the new polygons satisfy the following relation*:

$$\max(m_1, m_2) = n + 1.$$

Proof. Obvious. If one notices that the smallest possible number of vertices for a bounded convex n-gon is three and the unbounded convex n-gon with the smallest number of vertices is the 2-gon. The results of Lemmas 1 and 2 can be used to show that

$$\max(m_1, m_2) = n + 1. \quad \square$$

Theorem 1. *If a set of $n \geqslant 5$ lines defines a convex n-gon this n-gon is unique. If a set of $n \geqslant 3$ lines defines a bounded convex n-gon this bounded n-gon is unique.*

Proof. Obviously three straight lines define only one bounded convex 3-gon (triangle). Let $f_0(x, y) = 0$, $f_1(x, y) = 0$ and $f_2(x, y) = 0$ be the equations for lines L_0, L_1 and L_2 and A, B, C the vertices of the triangle (see Fig. 4(a)). All interior points of the triangle satisfy the following equations (see e.g. Kelly and Weiss, 1979, p. 134):

$$f_i(x, y) > 0 \quad \text{for } i = 0, 1, 2. \tag{3}$$

If this is not the case, one can always substitute $f_i(x, y) = -f_i(x, y)$ to make it happen. These three lines also define three unbounded 3-gons whose interior points satisfy the equations (Kelly and Weiss 1979, p. 134)

$$f_i(x, y) > 0, \qquad f_{i+1}(x, y) > 0,$$
$$f_{i+1}(x, y) < 0,$$
$$\text{for } i = 0, 1, 2. \tag{4}$$

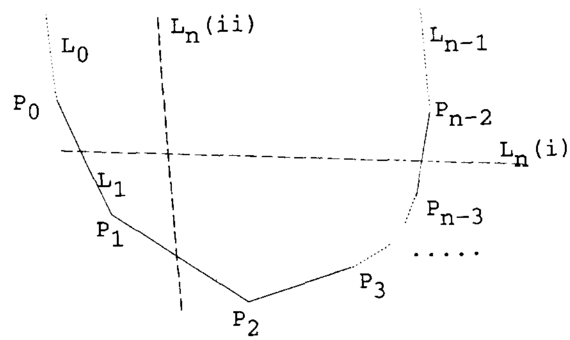

Fig. 3. An unbounded convex n-gon. When a straight line has a common point with the interior of the n-gon two cases exist: case (i) line L_n cuts the unbounded n-gon giving one bounded polygon and one unbounded convex polygon, case (ii) line L_n cuts the unbounded n-gon giving two unbounded polygons.

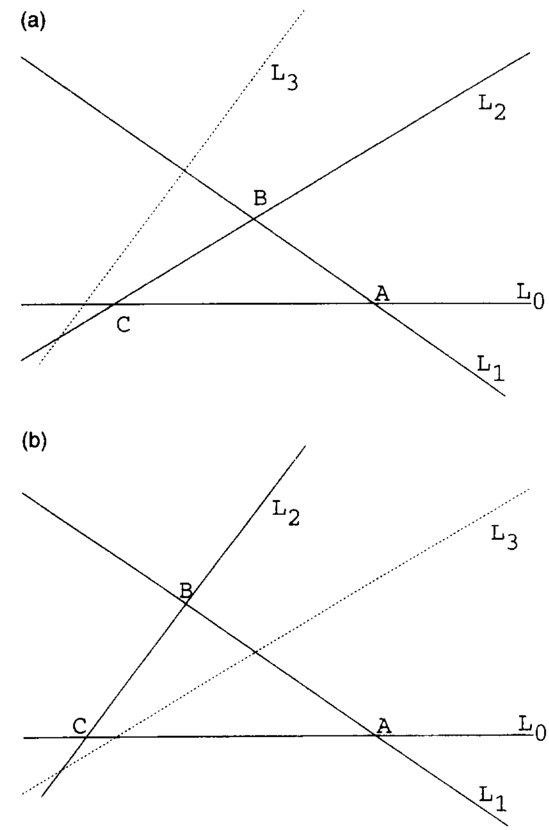

Fig. 4. (a) Figure showing the case where line L_3 does not cut the triangle ABC. The case is equivalent to (b) (i.e., interchange L_2 and L_3 and L_3 now cuts the single triangle defined by the set of lines $L_0 L_1 L_2$). (b) Figure showing the case where L_3 does cut the triangle ABC.

Whenever the subscript k, indicating a member of a set of n elements, exceeds the largest value an element can take (i.e., $n-1$ for a set of n elements when counting starts from zero), the element with subscript k mod(n) will be implied.

Suppose now, that a fourth line L_3 with equation $f_3(x, y) = 0$ is added to the set. Two cases exist: either L_3 cuts the triangle or it does not cut it (see Figs. 4(a) and 4(b)). It can be proved that the two cases are equivalent by just reordering the set of lines (if the line L_3 does not cut the triangle defined by lines $L_0 L_1 L_2$ then the line L_3 cuts the triangle defined by the lines $L_0 L_1 L_2$, this fact is illustrated in Figs. 4(a) and (4b)). Hence, we only need to check what happens to the unbounded 3-gons when a straight line cuts the triangle.

Suppose that $f_3(A, B) > 0$ and $f_3(C) < 0$ (see Fig. 5). Obviously L_3 has no common points with the boundary of the unbounded 3-gon $L_2 BAL_0$ (both its points of intersection with lines L_0 and L_2, D and E belong to the triangle ABC and from Eq. (3) we get $f_1(D, E) > 0$) and since $f_3(A, B) > 0$ it does not intersect the segment AB. Also, for the unbounded 3-gons $L_0 CBL_1$ and $L_1 ACL_2$, it is obvious that they have at least one common point with line L_3. Because line L_3 is not parallel with L_1 it has a common point with L_1, either in the ray AL_1 or in the ray BL_1. So a line L_3 that cuts the triangle defined by lines $L_0 L_1 L_2$ always gives an unbounded convex 4-gon by cutting one of the unbounded 3-gons created by the same lines; has one common point with the second unbounded 3-gon giving two unbounded 3-gons; and has no common point with the third unbounded 3-gon.

This means any set of four straight lines, where no pair is parallel and no three pass through the same point, gives one bounded convex 4-gon and one unbounded convex 4-gon (Fig. 5). For the case of two of the straight lines being parallel it can be shown that the set of four lines defines either one bounded convex 4-gon or two unbounded convex 4-gons. For the case of three of them passing through the same point or being parallel it is easy to show that they do not define a convex 4-gon. We will show now that five straight lines can only give one convex 5-gon, either bounded or unbounded.

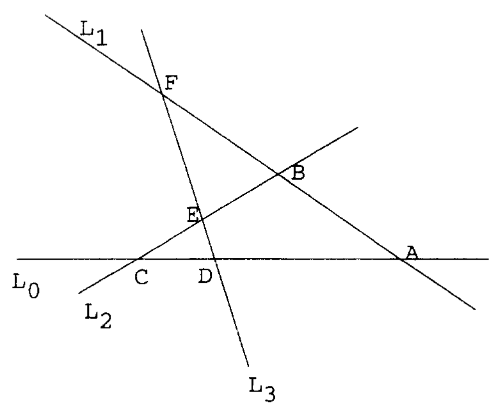

Fig. 5. Line L_3 cuts the triangle ABC giving a convex bounded 4-gon ($ABED$) while cutting the unbounded 3-gon $L_0 CBL_1$ giving an unbounded convex 4-gon ($L_0 CEFL_1$).

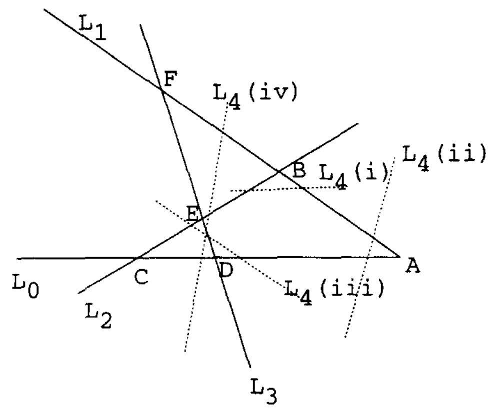

Fig. 6. Cases (i)–(iv) illustrate the necessary condition for a set of 5 straight lines to define a bounded convex 5-gon.

The bounded 4-gon is determined by the inequalities $f_i \geqslant 0$ for $i = 0, 1, 2, 3$ and the unbounded 4-gon by the inequalities $f_i \geqslant 0$ for $i = 0, 1$ and $f_i \leqslant 0$ for $i = 2, 3$. If line L_4 with equation $f_4(x, y) = 0$ cuts the 4-gon giving a 5-gon and a triangle, one of the following has to be satisfied:

(i) $f_4(B) < 0$, $f_4(E, D, A) > 0$ (see Fig. 6, line L_4 (i));
(ii) $f_4(A) < 0$, $f_4(B, E, D) > 0$ (see Fig. 6, line L_4 (ii));
(iii) $f_4(D) < 0$, $f_4(A, B, E) > 0$ (see Fig. 6, line L_4 (iii));
(iv) $f_4(E) < 0$, $f_4(D, A, B) > 0$ (see Fig. 6, line L_4 (iv)).

We will now show that if one of these four cases arises there is now way the unbounded 4-gon will be cut by the straight line L_4 to give a 5-gon.

First we study case (i) which is equivalent with case (iii). Because $f_4(B) < 0$ and $f_4(A) > 0$, we have $f_4(F) < 0$. Hence, line L_4 crosses line L_3 within the segment FE (see Lemmas 2 and 3). The only way the crossing of line L_4 to the unbounded 4-gon L_0CFL_1 can produce a 5-gon is by making a triangle one of whose vertices is either E or F. This implies that L_4 has a common point with either ray FL_1 or segment EC. Both cases are impossible because L_4 already has a common point with the line L_1 (within the segment AB) and line L_2 (within the segment BE).

Now we examine case (iv). Because $f_4(E) < 0$ and $f_4(D) > 0$ we have $f_4(F) < 0$. Using the same reasoning we can show that $f_4(C) < 0$. So line L_4 crosses lines L_0 and L_1 within the segments BF and CD, respectively. Also line L_4 crosses lines L_2 and L_3 within the segments EB and E_D. Obviously, none of the points of intersection of L_4 with lines L_i, $i = 0, 1, 2, 3$, belongs to the boundary of the unbounded convex 4-gon L_0CEFL_1. So line L_4 has no common points with the unbounded convex 4-gon. Case (ii) can be studied in the same way as case (iv).

Hence, if a line L_4 cuts a bounded convex 4-gon, defined by a set of four lines giving a bounded convex 5-gon it is guaranteed that the unbounded 4-gon which is defined by the same set of lines will not give a 5-gon. We can also prove the reverse. If the line cuts the unbounded convex 4-gon defined by the same set of four lines giving a 5-gon, it will not cut the bounded 4-gon defined by the same set of lines to give a 5-gon.

We proved that a set of 5 lines can only define one convex 5-gon, either bounded or unbounded. With the use of induction we will prove that for $n \geqslant 5$ a set of n lines can define only one convex n-gon.

(i) The case $n = 5$ has already been proved.
(ii) Suppose that if a set of $n = k > 5$ lines defines a convex k-gon, this k-gon is unique,
(iii) We will prove that this is also the case for $n = k + 1$. In other words we need to prove that given that a set of lines $L_0, L_1, \ldots, L_k$ defines a $(k + 1)$-gon this $(k + 1)$-gon is unique. Obviously any subset of k lines, say $L_0, L_1, \ldots, L_{k-1}$, defines at least one convex k-gon (bounded or unbounded). If this is not true then because of Lemma 3 it would be impossible for the set of $(k + 1)$ lines to define an $(k + 1)$-gon. By (ii) the k-gon is unique. The addition of one more line cuts it into two polygons one of which is by hypothesis a $(k + 1)$-gon. This $(k + 1)$-gon is unique because the k-gon is unique. □

We can summarize the results of Theorem 1 as follows (the assumption is that no three of the straight lines are parallel or have a common point).

(i) Three straight lines define one bounded convex 3-gon and three unbounded convex 3-gons.
(ii) Four straight lines define (if there is no pair of parallel lines) one bounded convex 4-gon and

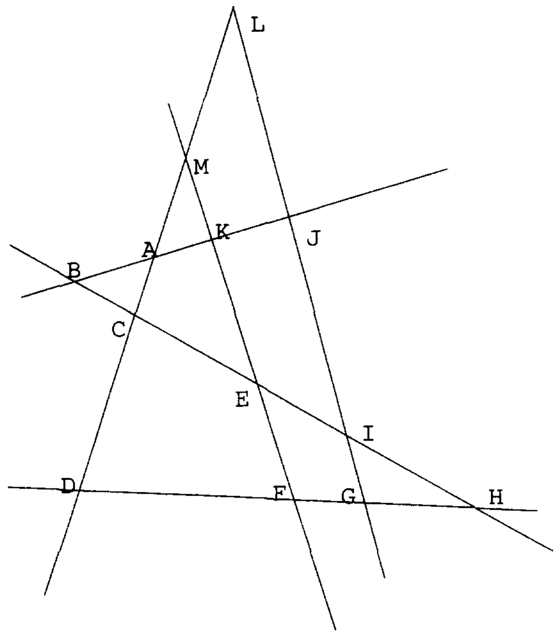

Fig. 7. An example where two non-convex polygons are defined by the same set of straight lines. Both non-convex hexagons *ADFEIJA* and *KECDGJK* are defined by the same set of straight lines (and therefore give the same peaks in the Hough space).

one unbounded convex 4-gon. If two of the lines are parallel, they define either two unbounded convex 4-gons or one bounded convex 4-gon.

(iii) n straight lines, where $n \geqslant 5$, can only define one convex n-gon (bounded or unbounded).

The previous theorem helps us to establish the following propositions.

Proposition 1. *Any bounded convex polygon is uniquely determined by the peaks of its Hough Transform.*

Proof. Obviously the n sides of the n-gon give n peaks in the Hough space (n straight lines). By hypothesis, these n peaks define a bounded convex n-gon. From Theorem 1 this n-gon is unique. □

Proposition 2. *For $n \geqslant 5$, a convex n-gon is uniquely defined by the peaks of its Hough Transform.*

Proof. The proof is similar to the proof of Proposition 1. □

4. Conclusions

We showed that any bounded convex polygon is uniquely determined by the peaks its vertices give in the Hough space. We also showed that a convex polygon with more than five sides is uniquely determined by the peaks it gives in the Hough space. Unfortunately as one can conclude by the example of Fig. 7 this is not the case for non-convex polygons.

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