REGULARIZATION IN TOMOGRAPHIC RECONSTRUCTION USING THRESHOLDING ESTIMATORS

Jérôme Kalifa, Andrew Laine

Department of Biomedical Engineering
Columbia University
New York, NY

Peter D. Esser

Department of Radiology
Columbia-Presbyterian Medical Center
New York, NY

ABSTRACT

In tomographic medical devices such as SPECT or PET cameras, image reconstruction is an unstable inverse problem, due to the presence of additive noise. A new family of regularization methods for reconstruction, based on a thresholding procedure in wavelet and wavelet packet decompositions, is studied. This approach is based on the fact that the decompositions provide a near-diagonalization of the inverse Radon transform and of the prior information on medical images. Corresponding algorithms have been developed for both 2-D and full 3-D reconstruction. These procedures are fast, non-iterative, flexible, and their performances outperform Filtered Back-Projection and iterative procedures such as OS-EM.

1. INTRODUCTION

We are interested in the problem of tomographic reconstruction of images from transmission data, which we call tomographic projections or *sinograms*. Although the work presented here has a wide range of applications for various tomographic devices, we will focus on medical images with SPECT and PET cameras.

A section of the object observed by the tomographic device is represented by a 2-D discrete image $f[n_1, n_2]$. An estimation of f must be computed with a tomographic reconstruction procedure from the sinograms produced by tomographic devices, denoted $Y[t, \theta]$, and defined as:

$$Y[t,\theta] = \mathcal{R}(f[n_1, n_2]) + W[t,\theta] \tag{1}$$

where $\{f[n_1, n_2]\}_{0 \leq n_1 \leq N_1 - 1,\, 0 \leq n_2 \leq N_2 - 1}$ is the observed image, W is an additive noise, and $\mathcal{R}$ is the discrete Radon transform which models the tomographic projection process. The discrete Radon transform is derived from its continuous version $\mathcal{R}_c$, which is equivalent to the X-ray transform in two dimensions and is defined as [1]

$$(\mathcal{R}_c f_c)(t,\theta) = \int_{\mathbb{R}} \int_{\mathbb{R}} f_c(x_1, x_2)\delta(x_1 \cos\theta + x_2 \sin\theta - t)dx_1 dx_2. \tag{2}$$

where $f_c(x_1, x_2) \in \mathbf{L^2}(\mathbb{R}^2)$, δ is the Dirac mass, $\theta \in [0, 2\pi)$, and $t \in \mathbb{R}$. There are several different ways to define the discrete Radon transform based on the continuous Radon transform [2]. Typically, a line integral along $x_1 \cos\theta + x_2 \sin\theta = t$ is approximated by a summation of the pixel values inside the strip $t - 1/2 \leq n_1 \cos\theta + n_2 \sin\theta < t + 1/2$.

The noise W is usually modelled as a Poisson, or sometimes Gaussian noise. However, since the tomographic projections Y are often processed to incorporate various corrections, such as attenuation correction, scatter correction, resolution correction or geometric correction, the resulting noise is also distorted and does not always comply with such prior models. The present approach can be adapted to different types of noise, including the case when there is no available statistical model for the noise.

A tomographic reconstruction procedure must incorporate the following steps: a *backprojection* may be viewed as the application of a discretized inverse Radon transform $\mathcal{R}^{-1}$ on the tomographic projections Y. This can be directly computed with a radial interpolation and a deconvolution to amplify the high frequency components of the tomographic projections Y in the direction of t. This deconvolution comes from the fact that the Radon transform is a smoothing transform. Consequently, backprojecting in the presence of additive noise is an *ill-posed* inverse problem: numerically speaking, a direct computation of $\mathcal{R}^{-1}Y = f + Z$ is contaminated by a large additive noise $Z = \mathcal{R}^{-1}W$, which means that a *regularization* has to be incorporated in the reconstruction procedure.

Current approaches for regularization in tomographic reconstruction can be summarized as follows:

1. Filtered Back-Projection (FBP) are linear filtering techniques in the Fourier space. FBP suffers from performance limitations due to the fact that the sinusoids of the Fourier basis are not adapted to represent spatially inhomogeneous data as found in medical images.
2. Iterative statistical model-based techniques are designed to implement Expectation-Maximization (EM)

Report Documentation Page

Report Date	Report Type	Dates Covered (from... to)
25OCT2001	N/A	-

Title and Subtitle Regularization in Tomographic Reconstruction Using Thresholding Estimators	**Contract Number**
	Grant Number
	Program Element Number
Author(s)	**Project Number**
	Task Number
	Work Unit Number
Performing Organization Name(s) and Address(es) Department of Biomedical Engineering Columbia University New York, New York	**Performing Organization Report Number**
Sponsoring/Monitoring Agency Name(s) and Address(es) US Army Research, Development & Standardization Group (UK) PSC 802 Box 15 FPO AE 09499-1500	**Sponsor/Monitor's Acronym(s)**
	Sponsor/Monitor's Report Number(s)
Distribution/Availability Statement Approved for public release, distribution unlimited	
Supplementary Notes Papers from the 23rd Annual International Conference of the IEEE Engineering in Medicine and Biology Society, October 25-28, 2001, held in Istanbul, Turkey. See also ADM001351 for entire conference on cd-rom.	
Abstract	
Subject Terms	
Report Classification unclassified	**Classification of this page** unclassified
Classification of Abstract unclassified	**Limitation of Abstract** UU
Number of Pages 4	

and Maximum A Posteriori (MAP) estimators [3, 4]. In some cases, these approaches can provide an improvement over FBP, but these estimators suffer from the following drawbacks:

- *Computation time.* Almost all the corresponding algorithms are too computer-intensive for clinical applications, with the exception of OS-EM [5] (an accelerated implementation of an EM estimator). In MAP methods, useful priors usually give local maxima, and the computational cost of relaxation methods is prohibitive.
- *Theoretical understanding and justification.* EM estimators lack theoretical foundations to understand and characterize the estimation error. Some MAP estimators are in some cases better understood, yet no optimality for a realistic model has been established.
- *Convergence.* EM estimators are ill-conditioned, in the sense that the corresponding iterative algorithms have to be stopped after a limited number of iterations. Beyond this critical number, the noise is magnified, and EM and OS-EM converge to a non-ML (Maximum Likelihood) solution.

This study aims to adress these limitations by building a family of estimation procedures which provide better numerical results, both metrically and perceptually, with a sound theoretical basis and for which the estimation error is understood and characterized. The estimators should also be implemented with fast and flexible algorithms.

2. THRESHOLDING ESTIMATORS

The estimation problem in (1) is also equivalent to the denoising problem

$$X = f + Z \tag{3}$$

where $X = \mathcal{R}^{-1}Y$ and $Z = \mathcal{R}^{-1}W$. If the noise Z was white, Donoho and Johnstone have established [6] that a thresholding estimator in a properly selected vector family $\mathcal{B} = \{g_{n_1,n_2}, g^*_{n_1,n_2}\}_{0\leq n_1\leq N_1-1,\,0\leq n_2\leq N_2-1}$, typically a wavelet basis, would be optimal to recover spatially inhomogeneous data as found in medical images. A thresholding estimator $\tilde{F}$ of f in $\mathcal{B}$ is defined as

$$\tilde{F} = \sum_{n_1,n_2} \rho_{n_1,n_2}\left(\langle X, g_{n_1,n_2}\rangle\right) g^*_{n_1,n_2} \ , \tag{4}$$

where ρ_{n_1,n_2} is a thresholding operator. In our situation, the choice of $\mathcal{B}$ does not only depend on the prior information on the object f, but also on the noise Z, whose behavior is very specific due to the fact that a backprojection has been applied on it.

The assumption underlying thresholding estimators is that each coefficient in the decomposition $\mathcal{B}$ can be estimated independently without a loss of performance. As a consequence, such estimators are efficient if the coefficients of the noise and of the object to be recovered are indeed nearly independent in $\mathcal{B}$. This means that $\mathcal{B}$ must provide a near-diagonalization of the noise Z and of the prior information on the image f.

The image f is a spatially inhomogeneous, piece-wise regular signal, which is compactly represented in a wavelet decomposition. To obtain a diagonal representation of the noise Z, we want to find a decomposition in which the inverse Radon transform is nearly diagonal. Since the inverse Radon transform is a Calderon-Zygmund operator [7], it is also nearly-diagonal in a wavelet basis.

These two properties of wavelet bases led Donoho [8] to suggest the use of thresholding estimators in wavelet bases for several linear inverse problems, including the inversion of the Radon transform. Donoho established the aymptotic optimality (minimax sense) of this approach, called Wavelet-Vaguelette Decomposition (WVD), and its superiority with respect to other approaches such as Filtered BackProjection. However, the WVD as studied by Donoho is a theoretical concept developped for continuous models, based on the assumption that the additive noise W is Gaussian white, and, despite numerical implementations and refinements by other researchers[9, 10], this technique is not as such adapted to the tomographic reconstruction of real medical images. In the meanwhile, Kalifa and Mallat [11] generalized Donoho's approach to adapt it to different types of decompositions, not restricted to wavelet bases.

Wavelet packet bases are other decompositions which can provide a compact representation of the observed image f, and since a wavelet packet transform provides a more accurate segmentation of the frequency domain than a wavelet transform, this improves the near-diagonalization of the noise Z. Besides, a wavelet packet basis can be adaptively chosen from a dictionary of different wavelet packet bases. We shall see in the next section that this enables us to optimize the choice of the wavelet packet transform to the specific type of observed image and to the nature of the backprojected noise Z.

3. CHOICE OF WAVELET PACKET DECOMPOSITION

Suppose we have a dictionary $\{\mathcal{B}^\gamma\}_\gamma$ of orthogonal wavelet packet bases. The empirical best basis $\mathcal{B}^{\gamma_1}$ for estimating f is obtained by minimizing an estimation of the final estimation error (risk), using the best basis algorithm of Coifman and Wickerhauser [12]. For example, the final estimation error can be estimated with a Stein Unbiased Risk Estimator (SURE) [13]. Another possibility is to take advantage of the fact that phantom images are usually available in tomography to model the observed images, and can be used as reference images to minimize the ideal projection risk, as defined in [6].

In both cases, a numerical model of the noise Z is computed. Depending on the fact that the original noise W is modelled as Gaussian or Poisson noise, or not modelled by a statistical prior, different procedures can be used. The numerical model of the noise Z is taken as input when computing the choice of the best basis, which is specifically adapted not only to the nature of the data but also to the nature of the noise Z. This adaptive approach enables us to improve greatly the performance of the decomposition, as compared to a classical wavelet basis.

4. RECONSTRUCTION ALGORITHM AND NUMERICAL RESULTS

The tomographic reconstruction algorithm is decomposed in the following steps:

1. Backprojection without regularization of the tomographic projections Y to obtain the backprojected image $X = f + Z$.
2. (Optional) Computation of the best wavelet packet basis $\mathcal{B}^{\gamma_1}$ optimized for the specific image to be restored. The best basis can be recomputed for each image, or can have been computed once and stored in advance, to save computation time. However the computation time of the best basis algorithm is very short.
3. Wavelet packet transform of the backprojected image X in the best basis $\mathcal{B}^{\gamma_1}$ to obtain the wavelet packet coefficients $\{\langle X, g_{n_1,n_2}\rangle\}_{n_1,n_2}$.
4. Thresholding on the wavelet packet coefficients.
5. Inverse wavelet packet transform of the thresholded coefficients to obtain the estimated image $\tilde{F}$.

The wavelet packet transform and its inverse are computed with fast filter bank algorithms of complexity $O(N)$ for signals of N samples[14].

Figure 1 compares preliminary numerical results computed on SPECT clinical data, using an OS-EM reconstruction, a Filtered Back-Projection and a wavelet-packet based reconstruction. The OS-EM reconstructed image is very smooth because the OS-EM algorithm has to be stopped after a limited number of iterations, otherwise the noise is strongly amplified and the algorithm converges to a very noisy reconstructed image. On the other hand, the FBP-reconstructed image is corrupted by a significant noise and artifacts, which cannot be reduced unless the reconstructed image becomes extremely smoothed. With the wavelet packet reconstruction algorithms, the amount of smoothness of a reconstructed image can be controlled precisely, while the noise is reduced significantly as compared to an image reconstructed with FBP or OS-EM. Besides, wavelet packet reconstruction methods are adapted to a broader range of images (different objects, low or high count) than OS-EM and FBP. The superiority of the wavelet-packet based reconstruction method over FBP and OS-EM is currently being established through ROC studies on various SPECT and PET data.

5. EXTENSION TO 3-D RECONSTRUCTION

So far, the wavelet packet reconstruction has been presented for 2-D reconstruction of slices. We now consider that we have a series of tomographic projections of N_3 translated 2-D slices of the observed object, i.e., that we have 3D data. When necessary, the tomographic projections have been transformed via rebinning techniques in order to obtain tomographic projections of 2-D slices: this approach is not necessary for SPECT images, but is increasingly common in 3-D PET imaging. It is useful to take advantage of the correlations of the signal in the transaxial direction to obtain a better discrimination between the information and the noise. In this case, a regularization is computed on the whole 3-D data, but the backprojections are still computed slice by slice, since the Radon transform is still only a bidimensional operator.

The 3-D thresholding reconstruction algorithm is summarized as follows. Details will be provided at the conference.

- Each 2-D slice is reconstructed and partly regularized using the wavelet packet reconstruction algorithm. The regularization is mild in order to preserve as much information as possible.

- The remaining noise (after the 2-D regularization) is nearly diagonalized in a wavelet decomposition. To take fully advantage of the 3-D information in the data, a 3-D dyadic wavelet transform is applied on the 3-D data, with wavelets adaptively oriented perpendicularly to the singularities of the signal. This directional selectivity enables us to maximize the correlation between the vectors of the wavelet family and the information of the signal. The efficiency of noise removal is thus greatly improved.

Acknowledgments

This work is supported in part by Siemens Medical Systems and the Whitaker Foundation. The authors would also like to thank François Gonczi and Sylvain Guedon for helping us implementing some of the algorithms.

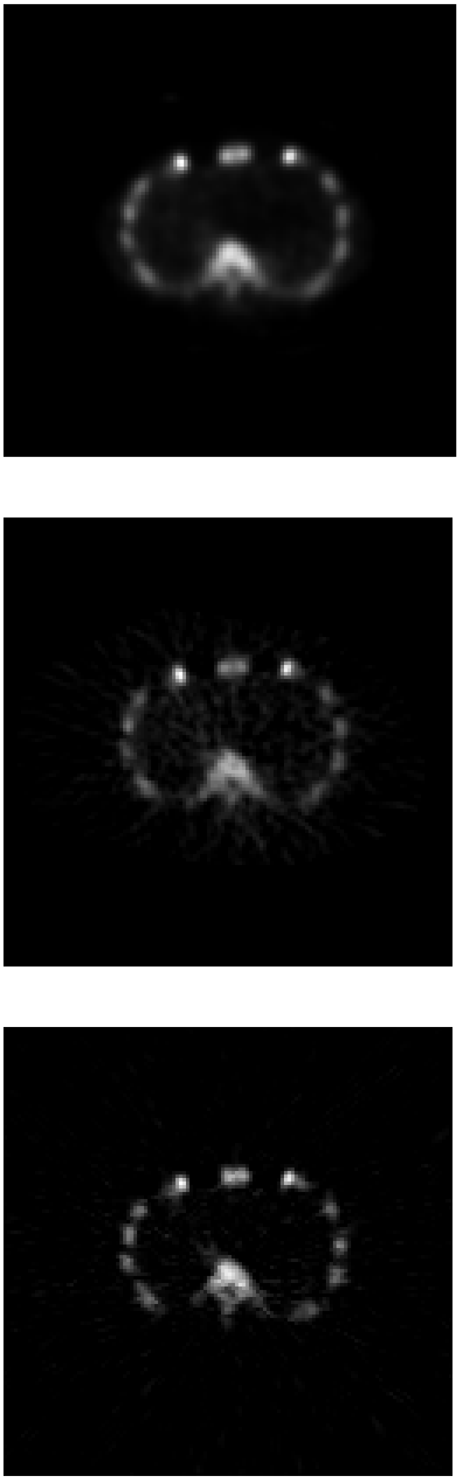

Fig. 1. SPECT image reconstructed with (a) OS-EM (b) Filtered BackProjection (c) thresholding in the wavelet packet basis.

6. REFERENCES

[1] S.R. Deans, *The Radon transform and some of its applications,* John Wiley and Sons, 1983.

[2] P. Toft, *The Radon Transform - Theory and Implementation*, Ph.D. thesis, Department of Mathematical Modelling, Technical University of Denmark, 1996, available at http://www.sslug.dk/~pto/cv.html.

[3] L.A. Shepp and Y. Vardi, "Maximum likelihood reconstruction for emission tomography," *Transactions on Medical Imaging*, pp. 113–122, 1982.

[4] G. McLachlan and T. Krishnan, *The EM Algorithm and Extensions*, New York:Wiley, 1997.

[5] H. Malcolm Hudson and Richard S. Larkin, "Accelerated image reconstruction using ordered subsets of projection data," *IEEE Transactions on Medical Imaging*, 1994.

[6] D. Donoho and I. Johnstone, "Ideal spatial adaptation via wavelet shrinkage," *Biometrika*, vol. 81, pp. 425–455, December 1994.

[7] Y. Meyer and R.R. Coifman, *Wavelets - Calderón-Zygmund and multilinear operators*, Cambridge Studies in Advanced Mathematics, 1997.

[8] D. Donoho, "Nonlinear solution of linear inverse problems by wavelet-vaguelette decompositions," *J. of Appl. and Comput. Harmonic Analysis*, vol. 2, no. 2, pp. 101–126, 1995.

[9] E. Kolaczyk, "A wavelet shrinkage approach to tomographic image reconstruction," *J. Amer. Statist. Assoc.*, vol. 91, pp. 1079–1090, 1996.

[10] N.M Lee and B. Lucier, "Wavelet methods for inverting the Radon transform with noisy data," *IEEE Trans. on IP, to appear*, 2000.

[11] J. Kalifa and S. Mallat, "Thresholding estimators for inverse problems and deconvolution," 1999, Submitted for publication, available at http://www.cmap.polytechnique.fr/~kalifa/.

[12] R. R. Coifman and M. V. Wickerhauser, "Entropy-based algorithms for best basis selection," *IEEE Trans. Info. Theory*, vol. 38, no. 2, pp. 713–718, March 1992.

[13] C. Stein, "Estimation of the mean of a multivariate normal distribution," *Annals of Statistics*, vol. 9, pp. 1135–1151, 1981.

[14] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, 2nd edition, 1999.